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On new sequences converging towards the Euler–Mascheroni constant

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ABSTRACT

We propose new sequences containing a modified logarithmic term which converge to the Euler–Mascheroni constant faster than sequences known from the literature.

1. Introduction

The Euler–Mascheroni constant $\gamma = 0.57721566490153286\ldots$ is defined in mathematics as the limit of the sequence

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$$

and it has numerous applications in many areas of pure and applied mathematics, such as analysis, theory of probability, special functions, applied statistics, or number theory. This constant is strangely ubiquitary and is probably the third most important constant in mathematics, after π and e. Nevertheless, the intimate nature of γ is very elusive; for example, it is not known if it is a rational number. A good part of its mystery comes from the fact that the known algorithms converging to γ are not very fast, at least, when they are compared to similar algorithms for π and e.

The sequence $(\gamma_n)_{n\geq 1}$ converges very slowly toward its limit, like n^{-1} and in consequence, many authors are preoccupied to improve its speed of convergence. We have

$$\frac{1}{2\,(n+1)} < \gamma_n - \gamma < \frac{1}{2n} \quad \text{(Young)}$$

(see, [1–3]) and other estimations can be found in [4–6].

Recently, DeTemple [7,8] had a clever idea to substantially improve the speed of convergence by introducing the sequence

$$R_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right).$$

It converges to γ like n^{-2}, since

$$\frac{1}{24\,(n+1)^2} < R_n - \gamma < \frac{1}{24n^2} \quad \text{(DeTemple)}.$$

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We define here a new class of sequences of the form

$$M_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln \frac{P(n)}{Q(n)},$$

where P, Q are polynomials having the leading coefficient equal to one and

$$\deg P - \deg Q = 1.$$

The classical sequence $(\gamma_n)_{n\geq 1}$ is obtained in case $P(n) = n$, $Q(n) = 1$, while DeTemple's sequence $(R_n)_{n\geq 1}$ is obtained for $P(n) = n + \frac{1}{2}$, $Q(n) = 1$.

Precisely, we introduce the sequences $(\nu_n)_{n\geq 1}$ and $(\mu_n)_{n\geq 1}$ by

$$\nu_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln \frac{n^2 + n + \frac{7}{24}}{n + \frac{1}{2}}$$

and

$$\mu_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln \frac{n^3 + \frac{3}{2}n^2 + \frac{227}{240}n + \frac{107}{480}}{n^2 + n + \frac{97}{240}}$$

which converge to γ like n^{-4}, respective n^{-6}, since

$$\lim_{n\to\infty} n^4 (\nu_n - \gamma) = -\frac{37}{5760} \quad \text{and} \quad \lim_{n\to\infty} n^6 (\mu_n - \gamma) = \frac{74\,381}{29\,030\,400}.$$

Our study is based on the following result, which gives a method for measuring the speed of convergence:

Lemma 1.1. *If $(\omega_n)_{n\geq 1}$ is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = l \in \mathbb{R}, \tag{1.1}$$

with $k > 1$, then there exists the limit:

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{l}{k-1}.$$

See [9] for the proof of this lemma and additional remarks. This result can be also used for constructing some asymptotic expansions, or to accelerate some convergences. See, e.g., [10–16,9].

We see from Lemma 1.1 that the speed of convergence of the sequence $(\omega_n)_{n\geq 1}$ is as higher as the value k satisfying (1.1) is greater.

2. The results

First we define the sequence

$$\nu_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln \frac{n^2 + an + b}{n + c}$$

and we are interested to find the values a, b, c which provide the fastest sequence $(\nu_n)_{n\geq 1}$. First

$$\nu_n - \nu_{n+1} = -\frac{1}{n+1} - \ln \frac{n^2 + an + b}{n + c} + \ln \frac{(n+1)^2 + a(n+1) + b}{n + 1 + c},$$

and we are concentrated to compute a limit of the form (1.1). To do this, we consider the representation in power series:

$$\begin{aligned}\nu_n - \nu_{n+1} = {} & \left(c - a + \frac{1}{2}\right)\frac{1}{n^2} - \left(-a^2 - a + c^2 + c + 2b + \frac{2}{3}\right)\frac{1}{n^3} \\ & - \left(a - 3b - c - 3ab + \frac{3}{2}a^2 + a^3 - \frac{3}{2}c^2 - c^3 - \frac{3}{4}\right)\frac{1}{n^4} \\ & + \left(a^4 + 2a^3 - 4a^2b + 2a^2 - 6ab + a + 2b^2 - 4b - c^4 - 2c^3 - 2c^2 - c - \frac{4}{5}\right)\frac{1}{n^5} + O\left(\frac{1}{n^6}\right).\end{aligned} \tag{2.1}$$

According to Lemma 1.1, we have three parameters a, b, c, which produce the fastest convergence of the sequence from (2.1)

$$\begin{cases} c - a + \dfrac{1}{2} = 0 \\ -a^2 - a + c^2 + c + 2b + \dfrac{2}{3} = 0 \\ a - 3b - c - 3ab + \dfrac{3}{2}a^2 + a^3 - \dfrac{3}{2}c^2 - c^3 - \dfrac{3}{4} = 0, \end{cases}$$

namely, if $a = 1$, $b = \frac{7}{24}$, $c = \frac{1}{2}$.

Using Lemma 1.1 and relation (2.1), we can state the following

Theorem 2.1. (i) *If $c - a + \frac{1}{2} \neq 0$, then the speed of convergence of the sequence $(\nu_n)_{n\geq 1}$ is n^{-1}, since*

$$\lim_{n\to\infty} n^2 (\nu_n - \nu_{n+1}) = c - a + \frac{1}{2} \quad \textit{and} \quad \lim_{n\to\infty} n (\nu_n - \gamma) = c - a + \frac{1}{2} \neq 0.$$

(ii) *If $c - a + \frac{1}{2} = 0$ and $-a^2 - a + c^2 + c + 2b + \frac{2}{3} \neq 0$, then the speed of convergence of the sequence $(\nu_n)_{n\geq 1}$ is n^{-2}, since*

$$\lim_{n\to\infty} n^3 (\nu_n - \nu_{n+1}) = -a^2 - a + c^2 + c + 2b + \frac{2}{3}$$

and consequently,

$$\lim_{n\to\infty} n^2 (\nu_n - \gamma) = \frac{1}{2}\left(-a^2 - a + c^2 + c + 2b + \frac{2}{3}\right) \neq 0.$$

(iii) *If $c - a + \frac{1}{2} = 0$, $-a^2 - a + c^2 + c + 2b + \frac{2}{3} = 0$ and $a - 3b - c - 3ab + \frac{3}{2}a^2 + a^3 - \frac{3}{2}c^2 - c^3 - \frac{3}{4} \neq 0$, then the speed of convergence of the sequence $(\nu_n)_{n\geq 1}$ is n^{-3}, since*

$$\lim_{n\to\infty} n^4 (\nu_n - \nu_{n+1}) = a - 3b - c - 3ab + \frac{3}{2}a^2 + a^3 - \frac{3}{2}c^2 - c^3 - \frac{3}{4}$$

and consequently,

$$\lim_{n\to\infty} n^3 (\nu_n - \gamma) = \frac{1}{3}\left(a - 3b - c - 3ab + \frac{3}{2}a^2 + a^3 - \frac{3}{2}c^2 - c^3 - \frac{3}{4}\right) \neq 0.$$

(iv) *If $c - a + \frac{1}{2} = 0$, $-a^2 - a + c^2 + c + 2b + \frac{2}{3} = 0$ and $a - 3b - c - 3ab + \frac{3}{2}a^2 + a^3 - \frac{3}{2}c^2 - c^3 - \frac{3}{4} = 0$, (equivalently with $a = 1$, $b = 7/24$, $c = 1/2$), then the speed of convergence of the sequence $(\nu_n)_{n\geq 1}$ is n^{-4}, since*

$$\lim_{n\to\infty} n^5 (\nu_n - \nu_{n+1}) = -\frac{37}{1440} \quad \textit{and} \quad \lim_{n\to\infty} n^4 (\nu_n - \gamma) = -\frac{37}{5760}.$$

For $a = 1$, $b = 7/24$, $c = 1/2$, the power series (2.1) becomes

$$\nu_n - \nu_{n+1} = -\frac{37}{1440n^5} + O\left(\frac{1}{n^6}\right),$$

which explains the assertion (iv) of Theorem 2.1.

Now we define the sequence

$$\mu_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln \frac{n^3 + dn^2 + fn + g}{n^2 + hn + k}$$

and we are interested to find the values d, f, g, h, k which provide the fastest sequence $(\mu_n)_{n\geq 1}$. As above, we consider the representation in power series:

$$\begin{aligned} \mu_n - \mu_{n+1} = {} & \left(h - d + \frac{1}{2}\right)\frac{1}{n^2} + \left(d^2 + d - h^2 - h - 2f + 2k - \frac{2}{3}\right)\frac{1}{n^3} \\ & - \left(d - 3f + 3g - h + 3k - 3df + 3hk + \frac{3}{2}d^2 + d^3 - \frac{3}{2}h^2 - h^3 - \frac{3}{4}\right)\frac{1}{n^4} \\ & - \Bigg(4f - d - 6g + h - 4k + 4d^2 f - 4h^2 k + 6df - 4dg \\ & - 6hk - 2d^2 - 2d^3 - d^4 - 2f^2 + 2h^2 + 2h^3 + h^4 + 2k^2 + \frac{4}{5}\Bigg)\frac{1}{n^5} \end{aligned}$$

$$+\left(5f-d-10g+h-5k-5df^2+10d^2f+5d^3f-5d^2g+5hk^2\right.$$
$$-10h^2k-5h^3k+10df-10dg+5fg-10hk-\frac{5}{2}d^2-\frac{10}{3}d^3-\frac{5}{2}d^4$$
$$\left.-d^5-5f^2+\frac{5}{2}h^2+\frac{10}{3}h^3+\frac{5}{2}h^4+h^5+5k^2+\frac{5}{6}\right)\frac{1}{n^6}+O\left(\frac{1}{n^7}\right). \tag{2.2}$$

According to Lemma 1.1, we see that the fastest sequence $(\mu_n)_{n\geq 1}$ is obtained in case when as many of the first coefficients of (2.2) is canceled. As we have now five parameters d, f, g, h, k, they produce the best result if and only if $\alpha=\beta=\delta=\eta=\sigma=0$, where

$$\alpha=h-d+\frac{1}{2}$$
$$\beta=d^2+d-h^2-h-2f+2k-\frac{2}{3}$$
$$\delta=-\left(d-3f+3g-h+3k-3df+3hk+\frac{3}{2}d^2+d^3-\frac{3}{2}h^2-h^3-\frac{3}{4}\right)$$
$$\eta=-\left(4f-d-6g+h-4k+4d^2f-4h^2k+6df-4dg\right.$$
$$\left.-6hk-2d^2-2d^3-d^4-2f^2+2h^2+2h^3+h^4+2k^2+\frac{4}{5}\right)$$
$$\sigma=\left(5f-d-10g+h-5k-5df^2+10d^2f+5d^3f-5d^2g+5hk^2\right.$$
$$-10h^2k-5h^3k+10df-10dg+5fg-10hk-\frac{5}{2}d^2-\frac{10}{3}d^3-\frac{5}{2}d^4$$
$$\left.-d^5-5f^2+\frac{5}{2}h^2+\frac{10}{3}h^3+\frac{5}{2}h^4+h^5+5k^2+\frac{5}{6}\right).$$

The solution is $d=\frac{3}{2}, f=\frac{227}{240}$, $g=\frac{107}{480}, h=1, k=\frac{97}{240}$, which produces the best sequence $(\mu_n)_{n\geq 1}$.

Using Lemma 1.1 and relation (2.2), we can state the following

Theorem 2.2. (i) *If $\alpha\neq 0$, then the speed of convergence of the sequence $(\mu_n)_{n\geq 1}$ is n^{-1}, since*

$$\lim_{n\to\infty} n^2\left(\mu_n-\mu_{n+1}\right)=\alpha \quad \textit{and} \quad \lim_{n\to\infty} n\left(\mu_n-\gamma\right)=\alpha\neq 0.$$

(ii) *If $\alpha=0$ and $\beta\neq 0$, then the speed of convergence of the sequence $(\mu_n)_{n\geq 1}$ is n^{-2}, since*

$$\lim_{n\to\infty} n^3\left(\mu_n-\mu_{n+1}\right)=\beta \quad \textit{and} \quad \lim_{n\to\infty} n^2\left(\mu_n-\gamma\right)=\beta/2\neq 0.$$

(iii) *If $\alpha=\beta=0$ and $\delta\neq 0$, then the speed of convergence of the sequence $(\mu_n)_{n\geq 1}$ is n^{-3}, since*

$$\lim_{n\to\infty} n^4\left(\mu_n-\mu_{n+1}\right)=\delta \quad \textit{and} \quad \lim_{n\to\infty} n^3\left(\mu_n-\gamma\right)=\delta/3\neq 0.$$

(iv) *If $\alpha=\beta=\delta=0$ and $\eta\neq 0$, then the speed of convergence of the sequence $(\mu_n)_{n\geq 1}$ is n^{-4}, since*

$$\lim_{n\to\infty} n^5\left(\mu_n-\mu_{n+1}\right)=\eta \quad \textit{and} \quad \lim_{n\to\infty} n^4\left(\mu_n-\gamma\right)=\eta/4\neq 0.$$

(v) *If $\alpha=\beta=\delta=\eta=0$ and $\sigma\neq 0$, then the speed of convergence of the sequence $(\mu_n)_{n\geq 1}$ is n^{-5}, since*

$$\lim_{n\to\infty} n^6\left(\mu_n-\mu_{n+1}\right)=\sigma \quad \textit{and} \quad \lim_{n\to\infty} n^5\left(\mu_n-\gamma\right)=\sigma/5\neq 0.$$

(vi) *If $\alpha=\beta=\delta=\eta=\sigma=0$ (equivalently with $d=\frac{3}{2}, f=\frac{227}{240}, g=\frac{107}{480}, h=1, k=\frac{97}{240}$), then the speed of convergence of the sequence $(\mu_n)_{n\geq 1}$ is n^{-6}, since*

$$\lim_{n\to\infty} n^7\left(\mu_n-\mu_{n+1}\right)=\frac{74\,381}{4\,838\,400} \quad \textit{and} \quad \lim_{n\to\infty} n^6\left(\mu_n-\gamma\right)=\frac{74\,381}{29\,030\,400}.$$

For $d=\frac{3}{2}, f=\frac{227}{240},\ g=\frac{107}{480},\ h=1,\ k=\frac{97}{240}$, the power series (2.2) becomes

$$\mu_n-\mu_{n+1}=\frac{74\,381}{4\,838\,400n^7}+O\left(\frac{1}{n^8}\right),$$

which explains the assertion (vi) of Theorem 2.2.

3. Concluding remarks

As least theoretically, further sequences of the form

$$M_n=1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln\frac{P\,(n)}{Q\,(n)}$$

can be defined, with $\deg P=k+1$, $\deg Q=k\geq 3$. As above,

$$M_n-M_{n+1}=-\frac{1}{n+1}-\ln\frac{P\,(n)\,Q\,(n+1)}{Q\,(n)\,P\,(n+1)} \tag{3.1}$$

and if we develop (3.1) in power series of n^{-1}, then the $2k+1$ coefficients of the polynomials P and Q are the unique solution of the system obtained by imposing that the first $2k+1$ coefficients of (3.1) vanish. In this case,

$$M_n-M_{n+1}=\frac{\theta}{n^{2k+3}}+O\left(\frac{1}{n^{2k+4}}\right),$$

with $\theta\neq 0$. From Lemma 1.1, $(M_n)_{n\geq 1}$ tends to zero as $n^{-(2k+2)}$, since

$$\lim_{n\to\infty} n^{2k+2}M_n=\frac{\theta}{2k+2}\neq 0.$$

Finally, we offer some numerical computations to prove the superiority of our sequences $(\nu_n)_{n\geq 1}$ and $(\mu_n)_{n\geq 1}$ over the classical sequence $(\gamma_n)_{n\geq 1}$ and the DeTemple sequence $(R_n)_{n\geq 1}$. As an example, we see that already μ_1 approximates γ with eight exact decimals.

n	$\gamma_n-\gamma$	$R_n-\gamma$	$\gamma-\nu_n$	$\mu_n-\gamma$
10	4.9167×10^{-2}	3.7733×10^{-4}	5.2565×10^{-7}	$\mathbf{1.8859\times 10^{-9}}$
50	9.9667×10^{-3}	1.6337×10^{-5}	9.8744×10^{-10}	$\mathbf{1.5438\times 10^{-13}}$
100	4.9917×10^{-3}	4.1252×10^{-6}	6.2964×10^{-11}	$\mathbf{2.4863\times 10^{-15}}$
300	1.6657×10^{-3}	4.6142×10^{-7}	7.8777×10^{-13}	$\mathbf{3.4796\times 10^{-18}}$

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